On the Descriptional Complexity of Limited Propagating Lindenmayer Systems

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We investigate the descriptional complexity of limited propagating Lindenmayer systems and their deterministic and tabled variants with respect to the number of rules and the number of symbols. We determine the decrease of complexity when the generative capacity is increased. For incomparable families, we give languages that can be described more efficiently in either of these families than in the other.

1 Introduction

Several generating devices for formal languages have been studied in the literature with respect to the size of their descriptions (e. g., [2]). For sequentially deriving grammars, the measures number of productions, number of nonterminal symbols, and number of all symbols have been investigated.

In 1968, Lindenmayer systems (L-systems) have been introduced ([4]). In order to model the development of organisms, these devices work in parallel (in one derivation step, not only one symbol is rewritten as in a sequential grammar but all symbols are rewritten). For L-systems, the number of tables, the number of active symbols, and the degree of nondeterminism have been studied as measures of complexity. In [1], the measures number of rules and number of symbols were introduced for L-systems.

Twenty years after the introduction of L-systems, a restricted variant of L-systems with a partially parallel derivation process has been proposed in [6]. In these so-called k-limited L-systems, only k occurrences of each symbol are replaced according to some rule. First results on the descriptional complexity of k-limited L-systems can be found in [3].

We continue this work and study the relations that were left open in [3] or that have not been optimal yet. In this paper, we confine ourselves to propagating limited systems.

2 Definitions

We assume that the reader is familiar with the basic concepts of formal language theory (see e. g. [5]). We recall here some notations used in the paper.

We denote the set of all positive integers by $\mathbb{N}$ and the set of all non-negative integers by $\mathbb{N}_0$.

For an alphabet V (a finite set of symbols), we denote by V^* the set of all words over V, by V^+ the set of all non-empty words over V, and by V^n for a natural number $n \in \mathbb{N}_0$ the set of all words which have the length n. We denote the empty word by λ, the length of a word w by $|w|$, and the number of occurrences of a letter a in a word w by $|w|_a$. Furthermore, we denote the cardinality of a set A by $|A|$.

Two sets X and Y are called incomparable, if neither $X \subseteq Y$ nor $Y \subseteq X$ holds. They are called disjoint if the intersection is empty.

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A tabled interactionless Lindenmayer system (L-system for short), abbreviated as T0L system, is a triple $G = (V, \mathcal{P}, \omega)$ where V is an alphabet, $\omega \in V^+$ is called the axiom, and $\mathcal{P}$ is a finite, non-empty set $\{ P_1, P_2, \ldots, P_n \}$ where P_i (called a table), for $1 \leq i \leq n$, is a finite subset of $V \times V^*$ such that there is at least one element $(a, w) \in P_i$ for each letter $a \in V$. The elements (a, w) in some table are called productions or rules and are written as $a \to w$.

A T0L system $G = (V, \mathcal{P}, \omega)$ is called an 0L system if $\mathcal{P}$ contains only one table. It is called a DT0L system if every table $\mathcal{P}$ contains only one rule for each letter in V and it is called a D0L system if $\mathcal{P}$ contains only one table and the table consists of only one rule for each letter in V.

Such an L-system is called propagating, if there is no erasing rule $a \to \lambda$ in the system (all rules have the form $a \to w$ with $a \in V$ and $w \in V^+$).

A word $v \in V^+$ directly derives a word $w \in V^*$ by a system G, written as $v \underset{G}{\Longrightarrow} w$ (we omit the index if it is clear from the context), if $v = x_1 x_2 \cdots x_m$ with $m \in \mathbb{N}$, $x_i \in V$ for $1 \leq i \leq m$ and $w = y_1 y_2 \cdots y_m$ with $y_i \in V^*$ for $1 \leq i \leq m$ such that the system G contains a table P which contains all the rules $x_i \to y_i$ for $1 \leq i \leq m$. Hence, in parallel, every letter of a word is replaced by a word according to the rules of a table. By $\overset{*}{\Longrightarrow}$, we denote the reflexive and transitive closure of $\Longrightarrow$. The language generated by a system G is defined as

$$L(G) = \left\{ z \;\middle|\; \omega \underset{G}{\overset{*}{\Longrightarrow}} z \right\}.$$

In [6], a limitation of the parallel rewriting was introduced. For a natural number $k \in \mathbb{N}$, a k-limited T0L system (shortly written as $k\ell$T0L system) is a quadruple $G = (V, \mathcal{P}, \omega, k)$ where $(V, \mathcal{P}, \omega)$ is a T0L system. In a k-limited system, exactly $\min\{k, |w|_a\}$ occurrences of any letter a in the word w under consideration are rewritten in a derivation step (hence, the number of occurrences of a letter that are replaced in each step is limited by k).

We only say a T0L system is limited (shortly written as ℓT0L system) if it is a k-limited system for some number $k \in \mathbb{N}$.

The class of all k-limited T0L systems is written as $k\ell$T0L. The restricted and propagating variants thereof are denoted by $k\ell$PD0L, $k\ell$P0L, $k\ell$PDT0L, $k\ell$PT0L, and without k if the limit is arbitrary. For a class X of L-systems, we write $\mathcal{L}(X)$ for the family of languages that is generated by an L-system from X.

As measures of descriptional complexity, we consider the number of rules and the number of symbols. For an L-system G over an alphabet V with tables $P_1, P_2, \ldots, P_n$ with $n \in \mathbb{N}$ and an axiom ω, we set

$$\mathrm{Prod}(G) = \sum_{i=1}^{n} |P_i| \quad \text{and} \quad \mathrm{Symb}(G) = |\omega| + \sum_{i=1}^{n} \sum_{a \to w \in P_i} (|w| + 2).$$

Let X be a class of L-systems. For a language $L \in \mathcal{L}(X)$, we set

$$\mathrm{Prod}_X(L) = \min\{ \mathrm{Prod}(G) \mid G \in X \text{ with } L(G) = L \} \text{ and}$$
$$\mathrm{Symb}_X(L) = \min\{ \mathrm{Symb}(G) \mid G \in X \text{ with } L(G) = L \}.$$

Hence, the complexity of a language L with respect to a class X of L-systems is the complexity of a smallest L-system $G \in X$ that generates the language L. If we extend a class X to a class Y then the complexity can only become smaller: If $X \subseteq Y$, then $K_X(L) \geq K_Y(L)$ for any language $L \in \mathcal{L}(X)$ and complexity measure $K \in \{ \mathrm{Prod}, \mathrm{Symb} \}$.

We now define the complexity relations considered in this paper. Let X and Y be two classes of L-systems such that the language families $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ are not disjoint and let $K \in \{ \mathrm{Prod}, \mathrm{Symb} \}$ be a complexity measure.

We write

- $X =^K Y$ if $K_X(L) = K_Y(L)$ holds for any language $L \in \mathcal{L}(X) \cap \mathcal{L}(Y)$ (the complexities are equal),
- $X >^K Y$ if there is a sequence of languages $L_m \in \mathcal{L}(X) \cap \mathcal{L}(Y)$ for $m \in \mathbb{N}$, such that $K_X(L_m) - K_Y(L_m) \geq c \cdot m$ for a constant $c \in \mathbb{N}$ (the difference of the complexities can be arbitrarily large),
- $X \gg^K Y$ if there is a sequence of languages $L_m \in \mathcal{L}(X) \cap \mathcal{L}(Y)$ for $m \in \mathbb{N}$, such that $\lim\limits_{m\to\infty} \frac{K_Y(L_m)}{K_X(L_m)} = 0$ (asymptotically, the complexity using X grows faster than using Y),
- $X \ggg^K Y$ if there is a sequence of languages $L_m \in \mathcal{L}(X) \cap \mathcal{L}(Y)$, $m \in \mathbb{N}$, and a constant $c \in \mathbb{N}$ such that $K_Y(L_m) \leq c$ and $K_X(L_m) \geq m$.

From these definitions, we obtain that $X \ggg^K Y$ implies $X \gg^K Y$ and that also $X \gg^K Y$ implies $X >^K Y$ for $K \in \{\text{Prod}, \text{Symb}\}$.

For each natural number c, there are only finitely many L-systems G (upto renaming the symbols) for which $\text{Symb}(G) \leq c$ holds. Hence, there is no class X of L-systems that generates infinitely many languages L_n with $\text{Symb}_X(L_n) \leq c$. Thus, there exist no two classes X and Y with the relation $Y \ggg^{\text{Symb}} X$.

In all cases throughout this paper, we obtain the relation $X \gg^{\text{Symb}} Y$ whenever we also obtain $X \ggg^{\text{Prod}} Y$. Then, we also shortly write $X \ggg Y$. Further, if two classes X and Y are in the same relation $\rhd$ with respect to both measures Prod and Symb, hence, $X \rhd^{\text{Prod}} Y$ and $X \rhd^{\text{Symb}} Y$ for a symbol $\rhd \in \{\gg, >, =\}$, then we write $X \rhd Y$.

3 On 1-limited systems

Regarding 1-limited propagating L-systems, the following hierachy is known ([3]).

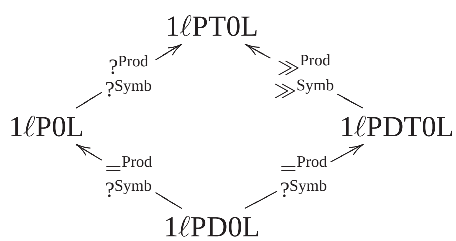

Figure 1: Relations for 1-limited systems

An arrow from a class X to a class Y with a label R is to be read as the relation XRY. If the label contains a question mark, then the relation was not given in [3]. In this section, we prove relations for all these cases and also relations between the classes 1ℓP0L and 1ℓPDT0L.

Theorem 3.1 *The relation* $1\ell\text{PD0L} =^{\text{Symb}} 1\ell\text{PDT0L}$ *holds.*

Proof. Let $G = (V, \{P\}, \omega, 1)$ be a 1ℓPD0L system which is minimal with respect to the number of symbols for the language $L = L(G)$ with $V = \{a_1, a_2, \ldots, a_n\}$ and $P = \{\, a_i \to w_{a_i} \mid 1 \leq i \leq n \,\}$. Further, let $H = (V, \{P_1, P_2, \ldots P_m\}, \omega_H, 1)$ be a minimal 1ℓPDT0L system for the language L.

Let Ω_n be the set of all words that are derived by G in n steps from the axiom:

$$\Omega_0 = \{\omega\},\ \Omega_n = \{\, w \mid \omega \Longrightarrow_G^n w \,\} = \{\, w \mid \text{there is } u \in \Omega_{n-1} \text{ with } u \Longrightarrow_G w \,\}.$$

For all $n \geq 0$, we have:

- The set Ω_n is not empty.
- All words in Ω_n contain the same number of letters for each letter of V (during the derivation, the same rules are applied – only at different positions). As a consequence, all words in Ω_n have the same length. Let it be denoted by l_n. The set of all occurring letters is denoted by α_n.

Since G is propagating, we have $l_0 \leq l_1 \leq l_2 \leq \cdots \leq l_i \leq \cdots$

For each word w_1, from which a word w_2 is derived by H in one step, there are words w_3 and w_4 such that the following holds:

- w_1 and w_3 belong to the same set Ω_p for a number $p \geq 0$,
- $w_3 \Longrightarrow_G w_4$ and
- $|w_4| \leq |w_2|$.

This implies

$$|w_4| = |w_3| + \sum_{\substack{i=1 \\ |w_3|_{a_i} > 0}}^{n} (|w_{a_i}| - 1)$$

(each letter a_i appearing in w_3 is replaced once by the corresponding word w_{a_i}; hence, $|w_{a_i}| - 1$ letters are added). Since $|w_2| \geq |w_4|$, we also have

$$|w_2| \geq |w_3| + \sum_{\substack{i=1 \\ |w_3|_{a_i} > 0}}^{n} (|w_{a_i}| - 1).$$

Since w_3 and w_1 belong to the same set Ω_p, we have $|w_3| = |w_1| = l_p$ and the words w_3 and w_1 consist of the same letters (the set of the appearing letters is α_p).

Hence,

$$|w_2| \geq |w_1| + s \text{ with } s = \sum_{\substack{i=1 \\ a_i \in \alpha_p}}^{n} (|w_{a_i}| - 1).$$

Let P_j be that table by which w_2 is derived from w_1 in H. Then we have, for the number $|P_j|$ of the symbols occurring in P_j,

$$|P_j| \geq s + \sum_{i=1}^{n} 3$$

(for each letter $a_i \in V$, there is a rule with at least three symbols; furthermore, the s new letters (the difference between w_2 and w_1) have to be generated and each rule is used at most once). In other words, we have

$$|P_j| \geq \sum_{\substack{i=1 \\ a_i \in \alpha_k}}^{n} (|w_{a_i}| + 2) + \sum_{\substack{i=1 \\ a_i \notin \alpha_k}}^{n} 3.$$

If P_j is applied to words from Ω_q and Ω_r for any $q \geq 0$ and $r \geq 0$, then this inequality holds for $p = q$ as well as for $p = r$. Thus,

$$|P_j| \geq \sum_{\substack{i=1 \\ a_i \in \alpha_q \cup \alpha_r}}^{n} (|w_{a_i}| + 2) + \sum_{\substack{i=1 \\ a_i \notin \alpha_q \cup \alpha_r}}^{n} 3.$$

Let A be the union of the sets α_p for those $p \geq 0$ for which P_j is applied to a word from Ω_p. Then

$$|P_j| \geq \sum_{\substack{i=1 \\ a_i \in A}}^{n} (|w_{a_i}|+2) + \sum_{\substack{i=1 \\ a_i \notin A}}^{n} 3.$$

Since P_j is applied to every word, we obtain $A = \bigcup\limits_{p \geq 0} \alpha_p = V$. Thus,

$$|P_j| \geq \sum_{i=1}^{n} (|w_{a_i}|+2) = \mathrm{Symb}(G) - |\omega|.$$

Together, this yields

$$\mathrm{Symb}(H) = |\omega_H| + \sum_{j=1}^{m} |P_j| \geq |\omega_H| + |P_1| \geq |\omega_H| + \mathrm{Symb}(G) - |\omega| = \mathrm{Symb}(G).$$

Since each 1ℓPD0L system is also a 1ℓPDT0L system, we have $\mathrm{Symb}(H) \leq \mathrm{Symb}(G)$ on the other hand. This yields the claim. □

The proof of the previous theorem can be changed such that H is a 1ℓP0L system (and P_j is the table of the system). Then we obtain the following result.

Corollary 3.2 *The relation* $1\ell\mathrm{PD0L} =^{\mathrm{Symb}} 1\ell\mathrm{P0L}$ *holds.*

Next, we prove relations between 1ℓP0L and 1ℓPT0L systems.

Theorem 3.3 *The relation* $1\ell\mathrm{P0L} \ggg 1\ell\mathrm{PT0L}$ *holds.*

Proof. Let $m \in \mathbb{N}$, $V = \{a,b,c,d,e\}$, and

$$L_m = \{e\} \cup \{\, a^n x_1 x_2 \cdots x_m d^n \mid n \geq 1,\ x_i \in \{b,c\},\ 1 \leq i \leq m \,\}.$$

The 1ℓPT0L system $G_m = (V, \{P_1, P_2\}, e, 1)$ with

$$P_1 = \{\, a \to a, b \to c, c \to c, d \to d, e \to ab^m d \,\} \text{ and}$$
$$P_2 = \{\, a \to aa, b \to b, c \to c, d \to dd, e \to e \,\}$$

generates the language L_m (the first table generates all words awd for $w \in \{b,c\}^*$ with $|w| = m$; the second table increases the number of occurrences of a and d). Since the complexities are $\mathrm{Prod}(G_m) = 10$ and $\mathrm{Symb}(G_m) = m+34$, we obtain $\mathrm{Prod}_{1\ell\mathrm{PT0L}}(L_m) \leq 10$ and $\mathrm{Symb}_{1\ell\mathrm{PT0L}}(L_m) \leq m+34$. Each language L_m is also generated by a 1ℓP0L system, for instance, by $G'_m = (V, \{P\}, e, 1)$ with

$$P = \{\, e \to awd \mid w \in \{b,c\}^*, |w| = m \,\} \cup \{\, a \to aa, b \to b, c \to c, d \to dd \,\}$$

(the first application of a rule yields all words awd for $w \in \{b,c\}^*$ with $|w| = m$; from the second application on, the number of occurrences of a and d is increased).

Now let $H_m = (V, \{P_m\}, \omega_m, 1)$ be a minimal 1ℓP0L system for L_m. Since H_m is propagating, ω_m is the shortest word of L_m: $\omega_m = e$. This word has to derive another word of L_m (otherwise only e is generated). Hence, P_m contains a rule $e \to ax_1x_2\cdots x_m d$ with $x_i \in \{b,c\}$ for $1 \leq i \leq m$ (if e derives only longer words, the words of length $m+2$ are not generated). Since the number of occurrences of a in the beginning of a word in L_m is unbounded, there must be a rule that increases the number. This

cannot be done by b, c, or d, because then an a would appear at a wrong position. Hence, it can only be done by a. If the rule for a contains other letters than a, then we obtain words that are not in L_m. Thus, P_m contains a rule $a \to a^i$ for an integer $i \geq 2$. If two different rules $a \to a^i$ and $a \to a^j$ exist, then two different words $a^i w'$ and $a^j w'$ could be generated but they are not both in L_m. Hence, there is only one rule $a \to a^i$ in P_m. The same argumentation holds for the rules of d. Hence, the only rule for d is $d \to d^i$ (if the rule would be $d \to d^j$ for a j different from i, then the word $ax_1x_2 \cdots x_m d$ would derive a word $a^{i-1}x'_1x'_2 \cdots x'_m d^{j-1} \notin L_m$). The only possible rules for b and c are $b \to b$ or $b \to c$ and $c \to c$ or $c \to b$, otherwise a word would be generated that does not belong to L_m.

Let $w = ax_1x_2 \cdots x_m d \in L_m$ be a word that is derived from e in one step. Then all words derived in one or more steps from w contain more than one a (because the only rule for a is $a \to a^i$ with $i \geq 2$). Hence, all words with only one a have to be derived directly from e. Hence, P_m contains at least all rules $e \to ax_1x_2 \cdots x_m d$ with $x_i \in \{b, c\}$ for $1 \leq i \leq m$. These are 2^m rules with $m+4$ symbols each.

Hence, $\mathrm{Prod}_{1\ell\mathrm{P0L}}(L_m) \geq 2^m$ and $\mathrm{Symb}_{1\ell\mathrm{P0L}}(L_m) \geq 2^m(m+4)$ which yields $1\ell\mathrm{P0L} \ggg^{\mathrm{Prod}} 1\ell\mathrm{PT0L}$ and $1\ell\mathrm{P0L} \gg^{\mathrm{Symb}} 1\ell\mathrm{PT0L}$. □

The two classes of 1ℓP0L systems and 1ℓPDT0L systems are incomparable. However, the language classes are not disjoint. Hence, we can also search for relations between incomparable classes.

Theorem 3.4 *The relations* $1\ell\mathrm{PDT0L} \gg^K 1\ell\mathrm{P0L}$ *for a complexity measure* $K \in \{\mathrm{Prod}, \mathrm{Symb}\}$ *as well as* $1\ell\mathrm{P0L} \ggg^{\mathrm{Prod}} 1\ell\mathrm{PDT0L}$ *and* $1\ell\mathrm{P0L} \gg^{\mathrm{Symb}} 1\ell\mathrm{PDT0L}$ *are valid.*

Proof. The first statement was shown in [3], although not explicitly mentioned (the 1ℓPT0L system used in the proof of the relation $1\ell\mathrm{PDT0L} \gg^K 1\ell\mathrm{PT0L}$ for $K \in \{Prod, \mathrm{Symb}\}$ is also a 1ℓP0L system).

The other two results follow from the proof of Theorem 3.3 because the 1ℓPT0L system used is also a 1ℓPDT0L system. □

The results for 1-limited propagating L-systems can be seen in the following figure.

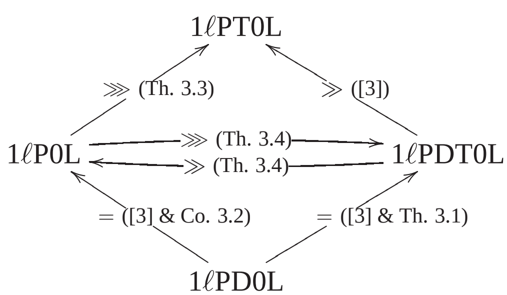

Figure 2: Results for 1-limited systems

In brackets behind a relation, you find a link to the corresponding proof.

If a sequence of languages L_m is generated by 1ℓP0L systems or 1ℓPT0L systems with a constant number of rules, then the languages L_m can also be generated by 1ℓPDT0L systems with a constant number of rules. As a consequence, all relations mentioned above are tight.

4 On higher limited systems

Let $k \geq 2$. Regarding k-limited propagating L-systems, the following hierachy is known ([3]).

An arrow from a class X to a class Y with a label R is to be read as the relation XRY. If the label contains a question mark, then the relation was not given in [3]. In this section, we give relations for these cases and also relations between the classes $k\ell$P0L and $k\ell$PDT0L.

$$\begin{tikzcd}
 & k\ell\mathrm{PT0L} & \\
k\ell\mathrm{P0L} \arrow[ur, "\ggg^{\mathrm{Prod}} \gg^{\mathrm{Symb}}"] & & k\ell\mathrm{PDT0L} \arrow[ul, "\ggg^{\mathrm{Prod}} \gg^{\mathrm{Symb}}"'] \\
 & k\ell\mathrm{PD0L} \arrow[ul, "=^{\mathrm{Prod}} ?^{\mathrm{Symb}}"] \arrow[ur, "=^{\mathrm{Prod}} ?^{\mathrm{Symb}}"'] &
\end{tikzcd}$$

Figure 3: Relations for k-limited systems

Theorem 4.1 *We have* $k\ell\mathrm{PD0L} =^{\mathrm{Symb}} k\ell\mathrm{PDT0L}$ *and* $k\ell\mathrm{PD0L} =^{\mathrm{Symb}} k\ell\mathrm{P0L}$ *for every* $k \geq 2$.

Proof. The proof is in both cases similar to the proof of Theorem 3.1. □

The two classes of $k\ell$P0L systems and $k\ell$PDT0L systems are incomparable. However, the classes of the generated languages are not disjoint. There are languages in the intersection that can be described more efficently by $k\ell$PDT0L systems than by $k\ell$P0L systems.

Theorem 4.2 *The relation* $k\ell\mathrm{P0L} \ggg k\ell\mathrm{PDT0L}$ *is valid for* $k \geq 2$.

Proof. We generalize the proof of Theorem 3.3. Let $k \geq 2$, $m \in \mathbb{N}$, and

$$L_m = \{e\} \cup \left\{ a^{kn} w d^{kn} \mid n \geq 1, w \in \{b,c\}^{km}, |w|_b = kj \text{ for } 0 \leq j \leq m \right\}.$$

This language is generated by the $k\ell$PDT0L system $G_m = (V, \{P_1, P_2\}, e, k)$ with $V = \{a,b,c,d,e\}$ and $P_1 = \left\{ a \to a, b \to c, c \to c, d \to d, e \to a^k b^{km} d^k \right\}$, $P_2 = \{ a \to aa, b \to b, c \to c, d \to dd, e \to e \}$. From this system, we obtain the relations $\mathrm{Prod}_{k\ell\mathrm{PDT0L}}(L_m) \leq 10$ and $\mathrm{Symb}_{k\ell\mathrm{PDT0L}}(L_m) \leq (m+2)k + 32$.

Each language L_m is also generated by a $k\ell$P0L system, for instance, by $G'_m = (V, \{P\}, e, k)$ with

$$P = \left\{ e \to a^k w d^k \;\middle|\; w \in \{b,c\}^{km}, |w|_b = kj \text{ for } 0 \leq j \leq m \right\} \cup \{ a \to aa, b \to b, c \to c, d \to dd \}.$$

By a similar argumentation as in the proof of Theorem 3.3, the rules for e being adopted to k, we obtain that a minimal $k\ell$P0L system contains at least all rules $e \to a^k w d^k$ where $w \in \{b,c\}^{km}$ and the number of bs in w is a multiple of k. Thus,

$$\mathrm{Prod}_{k\ell\mathrm{P0L}}(L_m) \geq 2^m \quad \text{and} \quad \mathrm{Symb}_{k\ell\mathrm{P0L}}(L_m) \geq 2^m((m+2)k+2)$$

which gives the relations $k\ell\mathrm{P0L} \ggg^{\mathrm{Prod}} k\ell\mathrm{PDT0L}$ and $k\ell\mathrm{P0L} \gg^{\mathrm{Symb}} k\ell\mathrm{PDT0L}$. □

The converse also holds. In the intersection $\mathcal{L}(k\ell\mathrm{P0L}) \cap \mathcal{L}(k\ell\mathrm{PDT0L})$, there are languages that can be described more efficently by $k\ell$P0L systems than by $k\ell$PDT0L systems.

Theorem 4.3 *The relation* $k\ell\mathrm{PDT0L} \ggg k\ell\mathrm{P0L}$ *is valid for* $k \geq 2$.

Proof. Let $k \geq 2$. Further, let $m \in \mathbb{N}$ be a natural number, $V = \{a,b,c\}$, and

$$L_m = \{c\} \cup \{ x_1 x_2 \cdots x_{km} \mid x_i \in \{a, bb\},\ 1 \leq i \leq km \}.$$

The $k\ell$P0L system $G_m = (V, \{P\}, c, k)$ with $P = \left\{ a \to a, a \to bb, b \to b, c \to a^{km} \right\}$ generates the language L_m (in each step, an arbitrary number j of as in a word with $0 \leq j \leq k$ can be choosen to

be changed to bb). As $\mathrm{Prod}(G_m) = 4$ and $\mathrm{Symb}(G_m) = km + 13$, we obtain $\mathrm{Prod}_{k\ell\mathrm{P0L}}(L_m) \leq 4$ and $\mathrm{Symb}_{k\ell\mathrm{P0L}}(L_m) \leq km + 13$.

Since $L_m \setminus \{c\}$ is finite, there is also a $k\ell$PDT0L system that generates the language (all words are derived from c).

Let H_m be a minimal $k\ell$PDT0L system. The axiom is c because it is the shortest word of L_m. For each word $z \in L_m \setminus \{c\}$, the equation $|z|_a + \frac{1}{2}|z|_b = km$ holds. Hence, the only possible rule for b in any table is $b \to b$; the only rules for a are $a \to a$ and $a \to bb$. The words $a^i bba^{km-1-i}$ $(0 \leq i \leq km-1)$ cannot be derived from other words of $L_m \setminus \{c\}$. Thus, H_m contains at least those km rules $c \to a^i bba^{km-1-i}$ and the $km \cdot (km+3)$ symbols involved.

Hence, $\mathrm{Prod}_{k\ell\mathrm{PDT0L}}(L_m) \geq km$ and $\mathrm{Symb}_{k\ell\mathrm{PDT0L}}(L_m) \geq km(km+3)$. This leads to the relations $k\ell\mathrm{PDT0L} \ggg^{\mathrm{Prod}} k\ell\mathrm{P0L}$ and $k\ell\mathrm{PDT0L} \gg^{\mathrm{Symb}} k\ell\mathrm{P0L}$. □

The results for k-limited propagating L-systems can be seen in the following figure.

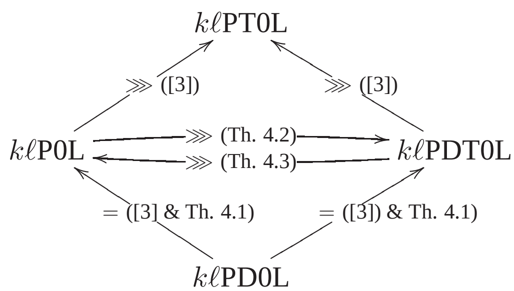

Figure 4: Results for k-limited systems

In brackets behind a relation, you find a link to the corresponding proof. All relations mentioned above are tight.

5 On arbitrarily limited systems

Regarding limited propagating L-systems, the following hierachy is known ([3]).

In this section, we prove relations for the open cases and also relations between the classes ℓP0L and ℓPDT0L.

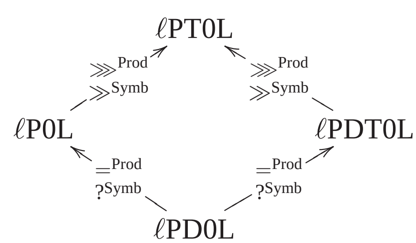

Figure 5: Relations for limited systems

The proofs for k-limited systems cannot directly be used because, for some k-limited system of one kind X of L-systems, there can be a minimal m-limited system of another kind Y with $m \neq k$ which has other properties than a minimal k-limited system of kind Y.

Theorem 5.1 *The relations* $\ell\mathrm{PD0L} \gg^{\mathrm{Symb}} X$ *hold for* $X \in \{\ell\mathrm{PDT0L}, \ell\mathrm{P0L}\}$.

Proof. Let $m \in \mathbb{N}$, $V = \{a, b, c, d\}$, and

$$L_m = \left\{ a^{m+im^2} b \mid i \geq 0 \right\} \cup \left\{ a^{m+im^2+1} c \mid i \geq 0 \right\} \cup \{ d \}.$$

Consider an ℓPD0L system $G_m = (V, \{P_{G_m}\}, \omega, k)$ generating the language L_m. The only derivation in G_m is

$$d \Longrightarrow a^m b \Longrightarrow a^{m+1} c \Longrightarrow a^{m+m^2} b \Longrightarrow a^{m+m^2+1} c \Longrightarrow \cdots$$
$$\cdots \Longrightarrow a^{m+im^2} b \Longrightarrow a^{m+im^2+1} c \Longrightarrow a^{m+(i+1)m^2} b \Longrightarrow a^{m+(i+1)m^2+1} c \Longrightarrow \cdots$$

because G_m is propagating. From this derivation, we obtain that G_m is minimal if $k = 1$ and the rule set P_{G_m} is $\left\{ a \to aa, b \to c, c \to a^{m^2-2} b, d \to a^m b \right\}$. Hence, the symbol complexity of L_m is $\mathrm{Symb}_{\ell\mathrm{PD0L}}(L_m) = m^2 + m + 12$.

The language L_m can also be generated by an m limited PDT0L system

$$H_m = (V, \{P_{m,1}, P_{m,2}\}, d, m)$$

with $P_{m,1} = \{ a \to aa^m, b \to b, c \to c, d \to a^m b \}$ and $P_{m,2} = \{ a \to a, b \to ac, c \to c, d \to d \}$. The first table derives from d the word $a^m b$ and from a word $a^p x \in L_m$ with $x \in \{b, c\}$ the word $a^{p+m^2} x$ (hence, every second word is generated). The second table derives from a word $a^{m+im^2} b$ with $i \geq 0$ the word $a^{m+im^2+1} c$ and leaves the other words unchanged.

Hence, we obtain for the symbol complexity of L_m with respect to ℓPDT0L systems

$$\mathrm{Symb}_{\ell\mathrm{PDT0L}}(L_m) \leq 2m + 26$$

which yields the relation $\ell\mathrm{PD0L} \gg^{\mathrm{Symb}} \ell\mathrm{PDT0L}$.

The language L_m can also be generated by an m limited P0L system $I_m = (V, \{P_{I_m}\}, d, m)$ with $P_{I_m} = \left\{ a \to aa^m, b \to b, c \to c, d \to a^m b, d \to a^{m+1} c \right\}$. In this system, we obtain the words $a^m b$ and $a^{m+1} c$ from d and then, by the other rules, from each word every second word.

Hence, we obtain $\mathrm{Symb}_{\ell\mathrm{P0L}}(L_m) \leq 3m + 17$ for the symbol complexity of L_m with respect to ℓP0L systems which yields the relation $\ell\mathrm{PD0L} \gg^{\mathrm{Symb}} \ell\mathrm{P0L}$. □

The two classes of ℓP0L systems and ℓPDT0L systems are incomparable. However, there are languages in the intersection of the classes that can be described more efficently by ℓPDT0L systems than by ℓP0L systems and vice versa.

Theorem 5.2 *The relations* $\ell\mathrm{P0L} \ggg \ell\mathrm{PDT0L}$ *and* $\ell\mathrm{PDT0L} \ggg \ell\mathrm{P0L}$ *hold.*

Proof. Let $m \in \mathbb{N}$, $V = \{a, b, c, d, e\}$, and

$$L_m = \{e\} \cup \{ a^n x_1 x_2 \cdots x_m d^n \mid n \geq 1,\ x_i \in \{b, c\},\ 1 \leq i \leq m \}.$$

Every language L_m is generatable by a 1ℓPDT0L system G_m – as shown in the proof of Theorem 3.3. From this proof, we obtain $\mathrm{Prod}_{\ell\mathrm{PDT0L}}(L_m) \leq 10$ and $\mathrm{Symb}_{\ell\mathrm{PDT0L}}(L_m) \leq m + 34$. Further, we know from that proof that each language L_m can also be generated by a 1ℓP0L system H_m. The argumentation on the minimal system does not depend on the limit k. Hence, any limited P0L system has at least 2^m rules and $2^m(m+4)$ symbols.

Thus, $\ell\mathrm{P0L} \ggg^{\mathrm{Prod}} \ell\mathrm{PDT0L}$ and $\ell\mathrm{P0L} \gg^{\mathrm{Symb}} \ell\mathrm{PDT0L}$.

To prove the other relation, let $m \in \mathbb{N}$, $V = \{a, b, c, d\}$, and

$$L_m = \{ d \} \cup \{ w \mid w = x_1 x_2 \cdots x_{2m},\ x_i \in \{a, bb, ccc\}, 1 \leq i \leq 2m, |w|_a = 2n,\ 0 \leq n \leq m \}.$$

The language L_m can be generated by a 2ℓP0L system $G_m = (V, \{P\}, d, 2)$ with the rule set being $P = \{ a \to bb, a \to ccc, b \to b, c \to c, d \to a^{2m} \}$. Hence, we obtain for the complexities

$$\mathrm{Prod}_{\ell\mathrm{P0L}}(L_m) \leq 5 \quad \text{and} \quad \mathrm{Symb}_{\ell\mathrm{P0L}}(L_m) \leq 2m + 18.$$

The language $L_m \setminus \{d\}$ is finite. Thus, L_m can be generated by a limited PDT0L system with a table for each rule $d \to w$ where $w \in L_m \setminus \{d\}$.

For each word $z \in L_m \setminus \{d\}$, the equation $|z|_a + \frac{1}{2}|z|_b + \frac{1}{3}|z|_c = 2m$ holds. Hence, the only possible rules in a minimal ℓPDT0L system H_m are $a \to a$, $a \to bb$, $a \to ccc$, $b \to b$, and $c \to c$.

If H_m is a 1ℓPDT0L system, then the word a^{2m} can derive the word bba^{2m-1} or $ccca^{2m-1}$ which are not in L_m unless the only rule for a is $a \to a$. But then every word $w \in L_m \setminus \{d\}$ has to be derived from d. This yields more than m rules and more than m^2 symbols.

If H_m is a $k\ell$PDT0L system for some $k \geq 2$, then any word $x_1 x_2 \cdots x_{2m}$ where one subword x_i is bb, one subword x_j is ccc, and all other subwords are a can only be derived from d (if bb or ccc is derived from a, then a second subword bb or ccc must exist, because there is only one rule for a and $k \geq 2$). This also yields more than m rules and more than m^2 symbols.

Hence, we obtain $\mathrm{Prod}_{\ell\mathrm{PDT0L}}(L_m) \geq m$ and $\mathrm{Symb}_{\ell\mathrm{PDT0L}}(L_m) \geq m^2$. This gives us the relations ℓPDT0L $\ggg^{\mathrm{Prod}}$ ℓP0L and ℓPDT0L $\gg^{\mathrm{Symb}}$ ℓP0L. □

The results for limited propagating L-systems can be seen in Figure 6. In brackets behind a relation, you find a link to the corresponding proof. All relations mentioned above are tight.

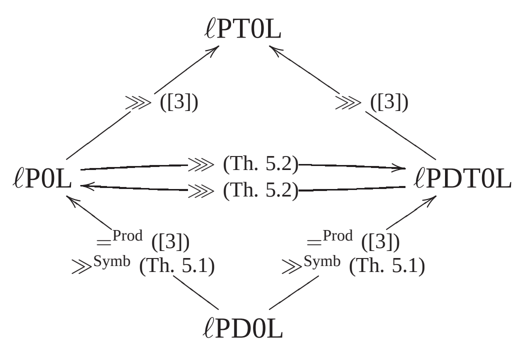

Figure 6: Results for limited systems

In the remaining part of this section, we investigate the relations between k-limited and arbitrarily limited propagating L-systems.

The relations that are already known (cf. [3]) are to be seen in Figure 7. These relations hold for $k \geq 1$.

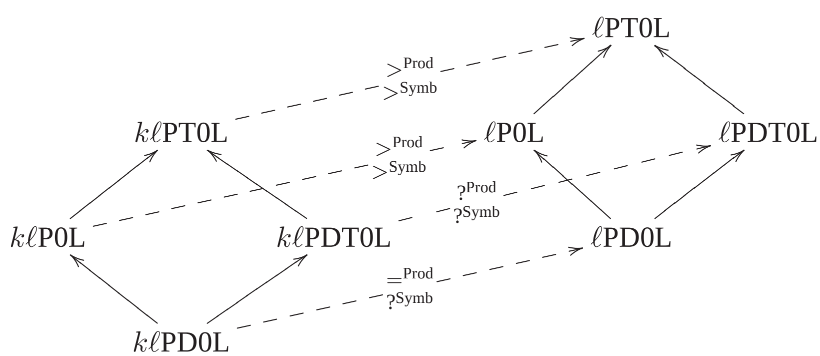

Figure 7: Relations between k-limited and limited propagating L-systems

For $k = 1$, the relations $k\ell$PDT0L $\gg^{K}$ ℓPDT0L for $K \in \{\mathrm{Prod}, \mathrm{Symb}\}$ are given in [3]. We now prove relations for the remaining cases and give stronger results for the existing ones.

Theorem 5.3 *The relation* $k\ell$PD0L $\gg^{\text{Symb}}$ ℓPD0L *holds for* $k \geq 1$.

Proof. Let $k \geq 1$. For $m \in \mathbb{N}$, let $L_m = \{a^{5mk(1+nm)} \mid n \geq 0\}$. Further, let $G_m = (\{a\},\{P\},\omega,k)$ be a $k\ell$PD0L system that generates the language L_m and that is minimal with respect to the number of symbols. Then we have $\omega = a^{5mk}$. From ω, the word a^{5mk+5m^2k} must be derived. Hence, the rule in P is $a \to aa^{5m^2}$. With this rule, the number of as is increased by $5m^2k$ in each step. We have $\text{Symb}_{k\ell\text{PD0L}}(L_m) = \text{Symb}(G_m) = 5mk + 5m^2 + 3$.

The system $H_m = (\{a\},\{a \to aa^m\},a^{5mk},5mk)$ for $m \geq 1$ is a limited PD0L system also generating L_m. We obtain $\text{Symb}_{\ell\text{PD0L}}(L_m) \leq 5mk + m + 3$. Hence, $k\ell$PD0L $\gg^{\text{Symb}}$ ℓPD0L for each $k \geq 1$. □

For the relations between the various types of k-limited and limited propagating L-systems, we give a results that covers them all.

Theorem 5.4 *The relation* $k\ell$PT0L $\ggg$ ℓPD0L *is valid for* $k \geq 1$.

Proof. Let $k \geq 1$ and $V = \{a,b,c\}$. For $m \in \mathbb{N}$, consider the language

$$L_m = \{c\} \cup \{w \mid w = x_1x_2\cdots x_{(k+1)m},\ x_i \in \{a,bb\}, 1 \leq i \leq (k+1)m, \\ |w|_a = j(k+1), 0 \leq j \leq m\}.$$

As shown in the proof of Theorem 4.3, the only possible rules for a and b are $a \to a$, $a \to bb$, and $b \to b$. The axiom of a minimal $k\ell$P0L system G_m is c and there is a rule $c \to a^{(k+1)m}$. If G_m contains the rule $a \to bb$, then words are derived that do not belong to L_m (e. g., words with exactly k subwords bb). Hence, the only rule for a is $a \to a$ in any table. Thus, all words of the set $L_m \setminus \{c\}$ have to be derived directly from c. This yields more than m rules and more than $m^2(k+1)$ symbols.

However, a $(k+1)$-limited PD0L system H_m with the rules $a \to bb$, $b \to b$, and $c \to a^{(k+1)m}$ also generates L_m but needs only three rules and $(k+1)m + 10$ symbols. This proves $k\ell$PT0L $\ggg^{\text{Prod}}$ ℓPD0L and $k\ell$PT0L $\gg^{\text{Symb}}$ ℓPD0L. □

From this result, we obtain the relations

$$k\ell X \ggg^{\text{Prod}} \ell X \quad \text{and} \quad k\ell X \gg^{\text{Symb}} \ell X$$

for all classes $X \in \{\text{P0L}, \text{PDT0L}, \text{PT0L}\}$.

Summarizing, we found and proved relations between all classes of limited propagating L-systems that were left open or that have not been considered in [3]. In some cases, we could improve the results in [3] regarding propagating systems. The relations we stated here are all tight.

Limited T0L systems that are not necessarily propagating have also been studied in [3]. Except only a few cases, all relations between the various classes have been found and have been proven to be tight. For some of the open questions, we can adopt results from the propagating case to the non-propagating case.

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